Lecture 02: Density of Primes

Density of primes

Let $\pi(x)$ be the number of primes $\leqslant x$. Equivalently, $\pi(x) := \sum_{p \leqslant x} 1$.
Progress towards the following result:

Theorem (Prime Number Theorem)

$$\pi(x) \sim \frac{x}{\log x}$$

The $\log(\cdot)$ is the natural logarithm.

- Adrien-Marie Legendre conjectured in 1808
- Independently proven by Jacques Hadamard and Charles Jean de la Vallée-Poussin in 1896

Infinitude of Primes

Theorem

There are infinitely many primes.

Proof 1:

- Suppose not
- Let $P = \{p_1, \ldots, p_t\}$ be the set of all primes
- Consider the number $n = \prod_{i \in [t]} p_i + 1$
- It is not divisible by any number in P
- Hence, contradiction

Something more is known:

Theorem (Dirichlet's Theorem)

For $a, d \in \mathbb{N}$ *and* $\gcd(a, d) = 1$, *there are infinitely many primes* $p \equiv a \mod d$

Proof 2:

- Suppose not and let p be the largest prime
- Consider the number $n = 2^p - 1$
- If n is a prime then it is $> p$
- If n is not a prime then consider a prime $q|n$
- That is, $n \equiv 0 \mod q$
- Alternately, $2^p \equiv 1 \mod q$
- Consider the multiplicative group $\mathbb{Z}_q^*$
- By Lagrange's Theorem, $p|q-1$
- That is $p < q$

Proof 3 (Lower Bounding $\pi(x)$):

- Let $n \leqslant x < n+1$
- $\log x \leqslant 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leqslant \sum_{m \in \mathbb{N}(n)} \frac{1}{m}$, where $\mathbb{N}(n)$ is the set of all natural numbers with all prime divisors $\leqslant n$
- Right hand side is identical to $\prod_{p \leqslant n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leqslant n} \frac{1}{1-p^{-1}} = \prod_{p \leqslant n} \frac{p}{p-1}$
- Let p_k be the k-th prime
- Note that $t/(t-1)$ is a decreasing function and $p_k \geqslant k+1$
- So, $\frac{p_k}{p_k - 1} \leqslant \frac{k+1}{k}$
- Therefore, $\prod_{p \leqslant n} \frac{p}{p-1} \leqslant \pi(x) + 1$
- Overall, $\pi(x) \geqslant \log x - 1$

Chebyshev's Estimates

Theorem (Chebyshev's Estimates)

$$\pi(x) = \Theta\left(\frac{x}{\log x}\right)$$

Lower Bound

- Let $N = \binom{2m}{m}$
- N is divisible only by prime number up to $2m$
- $\nu_p(N)$ be the maximum power of p in N
- $\nu_p(N) = \sum_{k \geqslant 1} \left(\lfloor 2m/p^k \rfloor - 2 \lfloor m/p^k \rfloor \right)$
- Note that each term in the right hand side is either 0 or 1
- For $k > \log(2m)/\log p$ the terms are 0
- Therefore, $\nu_p(N) \leqslant \log(2m)/\log p$
- Note that $N = \prod_{p \leqslant 2m} p^{\nu_p(N)}$
- So, $\log N = \sum_{p \leqslant 2m} \nu_p(N) \log p \leqslant \sum_{p \leqslant 2m} \log(2m) = \pi(2m)\log(2m)$
- Rearranging, $\pi(2m) \geqslant \log N / \log(2m) \geqslant \log(2^{2m}/2m)/\log(2m) \geqslant \left(\frac{1}{2}\log 2\right)(2m)$

Upper Bound

- Let $\vartheta(x) := \sum_{p \leqslant x} \log p$
- Note $\prod_{m < p \leqslant 2m} \leqslant N \leqslant 2^{2m}$
- Therefore, $\vartheta(2m) - \vartheta(m) \leqslant (2\log 2)m$
- For $m = 2^k$, we have $\vartheta(2m) \leqslant (2\log 2)(2m)$
- Now, $\pi(2m) = \sum_{p \leqslant 2m} 1 = \pi(\sqrt{2m}) + \sum_{\sqrt{2m} < p \leqslant 2m} 1 \leqslant \sqrt{2m} + 2\vartheta(2m)/\log(2m) \leqslant \sqrt{2m} + (4\log 2)(2m/\log(2m))$

Prime Number Theorem

Theorem

$$\pi(x) \sim \frac{x}{\log x}$$

- We know the following result: For $x \geqslant 59$, we have:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) < \pi(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right)$$

Theorem

For every n, there exists a prime number between $p \in [n, 2n)$.

- Implies $\pi(x) \geqslant \lg x$
- Prime number theorem implies this theorem
- Prime number theorem implies large number of primes in the range $[n, 2n)$
- Prime number theorem implies: For every $\varepsilon > 0$, there exists c, n_0 such that for all $n \geqslant n_0$ there are $c\frac{x}{\log x}$ primes in the range $[n, (1+\varepsilon)n)$

Estimating $\pi(x)$

- Logarithmic-integral function $\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}$
- Error in estimation: $|\pi(x) - \mathrm{li}(x)|$

Conjecture

$$|\pi(x) - \mathrm{li}(x)| < x^{1/2} \log x$$

Equivalent to Riemann hypothesis

Riemann Hypothesis

Reimann's zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Theorem (Euler's Identity)

For every real number $s > 1$, we have:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

Conjecture (Riemann Hypothesis)

For $s \in \mathbb{C}$, if $\zeta(s) = 0$ and $\mathrm{Re}(s) \in (0, 1)$, then $\mathrm{Re}(s) = 1/2$