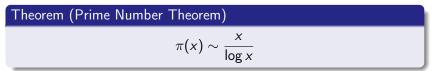
Lecture 02: Density of Primes

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Let $\pi(x)$ be the number of primes $\leq x$. Equivalently, $\pi(x) := \sum_{p \leq x} 1$. Progress towards the following result:



The $log(\cdot)$ is the natural logarithm.

- Adrien-Marie Legendre conjectured in 1808
- Independently proven by Jacques Hadamard and Charles Jean de la Vallée-Poussin in 1896

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Infinitude of Primes

Theorem

There are infinitely many primes.

Proof 1:

- Suppose not
- Let $P = \{p_1, \dots, p_t\}$ be the set of all primes
- Consider the number $n = \prod_{i \in [t]} p_i + 1$
- It is not divisible by any number in $\ensuremath{\boldsymbol{P}}$
- Hence, contradiction

Something more is known:

Theorem (Dirichlet's Theorem) For $a, d \in \mathbb{N}$ and gcd(a, d) = 1, there are infinitely many primes $p \equiv a \mod d$

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Infinitude of Primes

Proof 2:

- Suppose not and let p be the largest prime
- Consider the number $n = 2^p 1$
- If n is a prime then it is > p
- If n is not a prime then consider a prime q|n
- That is, $n \equiv 0 \mod q$
- Alternately, $2^p \equiv 1 \mod q$
- Consider the multiplicative group \mathbb{Z}_q^*
- By Lagrange's Theorem, p|q-1
- That is p < q

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Proof 3 (Lower Bounding $\pi(x)$):

- Let $n \leqslant x < n+1$
- $\log x \leq 1 + \frac{1}{2} + \dots + \frac{1}{n} \leq \sum_{m \in \mathbb{N}(n)} \frac{1}{m}$, where $\mathbb{N}(n)$ is the set of all natural numbers with all prime divisors $\leq n$
- Right hand side is identical to $\prod_{p \leqslant n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right) = \prod_{p \leqslant n} \frac{1}{1 - p^{-1}} = \prod_{p \leqslant n} \frac{p}{p - 1}$
- Let p_k be the k-th prime
- Note that t/(t-1) is a decreasing function and $p_k \geqslant k+1$
- So, $\frac{p_k}{p_k-1}\leqslant \frac{k+1}{k}$
- Therefore, $\prod_{p \leqslant n} \frac{p}{p-1} \leqslant \pi(x) + 1$
- Overall, $\pi(x) \ge \log x 1$

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Theorem (Chebyshev's Estimates)

$$\pi(x) = \Theta\left(\frac{x}{\log x}\right)$$

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Lower Bound

• Let
$$N = \begin{pmatrix} 2m \\ m \end{pmatrix}$$

- N is divisible only by prime number up to 2m
- $\nu_p(N)$ be the maximum power of p in N

•
$$\nu_p(N) = \sum_{k \ge 1} \left(\lfloor 2m/p^k \rfloor - 2 \lfloor m/p^k \rfloor \right)$$

- Note that each term in the right hand side is either 0 or 1
- For $k > \log(2m) / \log p$ the terms are 0
- Therefore, $u_p(N) \leqslant \log(2m) / \log p$
- Note that $N = \prod_{p \leqslant 2m} p^{\nu_p(N)}$
- So, $\log N = \sum_{p \leq 2m} \nu_p(N) \log p \leq \sum_{p \leq 2m} \log(2m) = \pi(2m) \log(2m)$
- Rearranging, $\pi(2m) \ge \log N / \log(2m) \ge \log(2^{2m}/2m) / \log(2m) \ge (\frac{1}{2}\log 2) (2m)$

• Let
$$\vartheta(x) := \sum_{p \leq x} \log p$$

• Note $\prod_{m
• Therefore, $\vartheta(2m) - \vartheta(m) \leq (2 \log 2)m$
• For $m = 2^k$, we have $\vartheta(2m) \leq (2 \log 2)(2m)$
• Now, $\pi(2m) = \sum_{p \leq 2m} 1 = \pi(\sqrt{2m}) + \sum_{\sqrt{2m}$$

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Theorem $\pi(x) \sim \frac{x}{\log x}$

• We know the following result: For $x \ge 59$, we have:

$$\frac{x}{\log x} \left(1 + \frac{1}{2\log x}\right) < \pi(x) < \frac{x}{\log x} \left(1 + \frac{3}{2\log x}\right)$$

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Theorem

For every n, there exists a prime number between $p \in [n, 2n)$.

- Implies $\pi(x) \ge \lg x$
- Prime number theorem implies this theorem
- Prime number theorem implies large number of primes in the range [*n*, 2*n*)
- Prime number theorem implies: For every $\varepsilon > 0$, there exists c, n_0 such that for all $n \ge n_0$ there are $c \frac{x}{\log x}$ primes in the range $[n, (1 + \varepsilon)n)$

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- Logarithmic-integral function $li(x) := \int_2^x \frac{dt}{\log t}$
- Error in estimation: $|\pi(x) Ii(x)|$

Conjecture

$$|\pi(x) - \mathsf{li}(x)| < x^{1/2} \log x$$

Equivalent to Riemann hypothesis

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Riemann Hypothesis

Reimann's zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Theorem (Euler's Identity)

For every real number s > 1, we have:

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}$$

Conjecture (Riemann Hypothesis)

For $s \in \mathbb{C}$, if $\zeta(s) = 0$ and $\mathsf{Re}(s) \in (0,1)$, then $\mathsf{Re}(s) = 1/2$

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